Geometrical Theory of Whispering-Gallery Modes

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Abstract—Using a quasi-classical approach, rather precise analytical approximations for the eigenfrequencies of whispering-gallery modes (WGMs) in convex axisymmetric bodies may be found. We use the eikonal method to analyze the limits of precision of quasi-classical approximation using, as a practical example, a spheroidal dielectric cavity. The series obtained for the calculation of eigenfrequencies is compared with the known series for a dielectric sphere, and with numerical calculations. We show how geometrical interpretation allows expansion of the method on arbitrarily-shaped axisymmetric bodies.

Index Terms—Eikonal, microspheres, spheroid, whispering-gallery modes (WGMs).

I. INTRODUCTION

SUBMILLIMETER size optical microspheres made of fused silica with whispering-gallery modes (WGMs) [1] can have an extremely high quality-factor, up to $10^{10}$, that makes them promising devices for applications in optoelectronics and experimental physics. Historically, Richtmyer [2] was the first to suggest that WGMs in axisymmetric dielectric body should have a very high quality-factor. He examined the cases of spheres and toruses. However, only recent breakthroughs in technology in several labs allowed producing not only spherical, and fused silica, but spheroidal, toroidal [3], [4], or even arbitrary form axisymmetrical optical microcavities from crystalline materials, thereby preserving or even increasing a high quality factor [5]. Especially interesting are devices manufactured of nonlinear optical crystals. Microresonators of this type can be used as high-finesse cavities for laser stabilization, frequency discriminating, and comparing them with other approximations; and 3) in the limit of zero eccentricity, a spheroid turns to a sphere for which the exact solution and series over $l$ up to $l^{-8/3}$ is known [9].

The Helmholtz vector equation is unseparable in spheroidal coordinates, and no vector harmonics tangential to the surface of a spheroid can be build. That is why there are no pure transverse electric (TE) or transverse magnetic (TM) modes in spheroids, but only in hybrid ones. Different methods of separation of variables (SVM) using series expansions with either spheroidal or spherical functions have been proposed [11]–[13]. Unfortunately, they lead to extremely bulky infinite sets of equations which can be solved numerically only in the simplest cases, and the convergence is not proved. An exact characteristic equation for the eigenfrequencies in a dielectric spheroid was suggested [14] without proof that, if real, could significantly ease the task of finding eigenfrequencies. However, we can not confirm this claim, as this characteristic equation contradicts limiting cases with the known solutions, i.e., ideal sphere and axisymmetrical oscillations in a spheroid with perfectly conducting walls [15].

Nevertheless, in the case of whispering gallery modes adjacent to equatorial plane, the energy is mostly concentrated in tangential, or normal to the surface, electric components that can be treated as quasi-TE or quasi-TM modes, and analyzed with good approximation using scalar wave equations.

Using quasi-classical method, we obtain the following practical approximation for the eigenfrequencies of WGMs in a spheroid:

$$nka = l - \alpha_q \left( \frac{l}{2} \right)^{1/3} + \frac{2(a - b) + a}{2b} - \frac{\chi_n}{\sqrt{n^2 - 1}} + \frac{3\alpha_q^2}{20} \left( \frac{l}{2} \right)^{-1/3} - \alpha_q \left( \frac{2(a^3 - b^3) + a^3}{b^3} + \frac{2\chi_n(2b^2 - 3n^2)}{(n^2 - 1)^{3/2}} \right) \left( \frac{l}{2} \right)^{-2/3} + O(l^{-1})$$

where $\alpha_q$ are equatorial and polar semiaxises, $k$ is wavenumber, $l \gg 1, p = l - |n| = 0, 1, 2, \ldots$ and $q = 1, 2, 3, \ldots$ are integer mode indices, $\alpha_q$ are negative $q$th zeroes of the Airy
function, \( n \) is the refraction index of a spheroid, and \( \chi = 1 \) for quasi-TE and \( \chi = 1/n^2 \) for quasi-TM modes.

II. SPHEROIDAL COORDINATE SYSTEM

There are several equivalent ways to introduce prolate and oblate spheroidal coordinate systems \((\xi, \eta, \phi)\). The following widely used system of coordinates allows us to analyze prolate and oblate geometries simultaneously:

\[
\begin{align*}
x &= \frac{d}{2}[(\xi^2 - s)(1 - \eta^2)]^{1/2} \cos(\phi) \\
y &= \frac{d}{2}[(\xi^2 - s)(1 - \eta^2)]^{1/2} \sin(\phi) \\
z &= \frac{d}{2} \xi \eta
\end{align*}
\]

where sign variable \( s \) is equal to 1 for the prolate geometry with \( \xi \in [1, \infty) \) determining spheroids, and \( \eta \in [-1, 1] \) describing two-sheeted hyperboloids of revolution (Fig. 1, right). Consequently, \( s = -1 \) gives oblate spheroids for \( \xi \in [0, \infty) \) and one-sheeted hyperboloids of revolution (Fig. 1, right). \( d/2 \) is the semidistance between focal points. We are interested in the modes inside a spheroid adjacent to its surface in the equatorial plane. It is convenient to designate a semiaxis in this plane as \( a \), and in the \( z \)-axis of rotational symmetry of the body as \( b \). In this case, \( d^2/4 = s(b^2 - a^2) \), and eccentricity \( \varepsilon = \sqrt{1 - (a/b)^2} \).

The scale factors for this system are the following:

\[
\begin{align*}
h_\xi &= \frac{d}{2} \left( \frac{\xi^2 - s \eta^2}{\xi^2 - s} \right)^{1/2} \\
h_\eta &= \frac{d}{2} \left( \frac{\xi^2 - s \eta^2}{1 - \eta^2} \right)^{1/2} \\
h_\phi &= \frac{d}{2}[(\xi^2 - s)(1 - \eta^2)]^{1/2}.
\end{align*}
\]

The scalar Helmholtz differential equation is separable

\[
\Delta \Phi + k^2 \Phi = 0
\]

\[
\frac{\partial}{\partial \xi} (\xi^2 - s) \frac{\partial \Phi}{\partial \xi} + \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial \Phi}{\partial \eta} + \left(c^2(\xi^2 + \eta^2) - \frac{m^2}{1 - \eta^2} - s \frac{m^2}{\xi^2 - s} \right), \quad \Phi = 0
\]

where \( c = kd/2 \). The solution is \( \Phi = R_{m,l}(c, \xi)S_{m,l}(c, \eta)e^{im \phi} \) where radial and angular functions are determined by the following equations:

\[
\frac{\partial}{\partial \xi} (\xi^2 - s) \frac{\partial R}{\partial \xi} - \left(\lambda_{m,l} - c^2 \xi^2 + s \frac{m^2}{\xi^2 - s} \right) R = 0 \quad (6)
\]

\[
\frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial S}{\partial \eta} + \left(\lambda_{m,l} - c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right) S = 0. \quad (7)
\]

Here, \( \lambda_{m,l} \) is the separation constant of the equations, which should be independently determined, and it is a function on \( m, l, \) and \( c \). With substitution \( \xi = 2r/d \), the first equation transforms to the equation for the spherical Bessel function \( j_l(kr) \) if \( d/2 \to 0 \), in which case the second equation immediately turns to the equation for the associated Legendre polynomials \( P^l_m(\eta) \) with \( \lambda_{m,l} = l(l + 1) \). That is why spheroidal functions are frequently analyzed as a decomposition over these spherical functions.

The calculation of spheroidal functions and of \( \lambda_{m,l} \) is not a trivial task [19], [20]. The approximation of spheroidal functions and their zeroes may seem more straightforward for the calculation of eigenfrequencies of spheroids; however, we found that another approach, which we develop as follows, gives better results, and may be easily generalized to other geometries.

III. EIKONAL APPROXIMATION IN SPHEROID

The eikonal approximation is a powerful method for solving optical problems in inhomogeneous media where the scale of the variations is much larger than the wavelength. It was shown by Keller and Rubinow [7] that it can also be applied to eigenfrequency problems, and that it has very clear quasi-classical ray interpretation. It is important that this quasi-classical ray interpretation, requiring simple calculation of the ray paths along the geodesic surfaces and application of phase equality (quantum) conditions, gives precisely the same equations as the eikonal equations. Eikonal equations allow us to more easily obtain not only eigenfrequencies, but field distribution as well.

In the eikonal approximation, the solution of the Helmholtz scalar equation is found as superposition of straight rays

\[
u(r) = A(r)e^{ik_0 S(r)}.
\]

The first-order approximation for the phase function \( S \), called eikonal, is determined by the following equation:

\[
(\nabla S)^2 = \epsilon(r)
\]

where \( \epsilon \) is optical susceptibility. For our problem of searching for eigenfrequencies, \( \epsilon \) does not depend on coordinates \( \epsilon = n^2 \) inside the cavity and \( \epsilon = 1 \) outside. The eikonal in the external area for dielectric cavities can be found as complex rays and stitched on the boundary. Alternatively, the ray method of Keller and Rubinow [7], [8], [21] can be extended for WGMs in dielectric bodies in a more simple way [22]. To do so, we must account for an additional phase shift on the dielectric boundary.
The Fresnel amplitude coefficient of reflection \([23]\)
\[
\mathcal{R} = \frac{\chi \cos \theta - i \sqrt{n^2 \sin^2 \theta - 1}}{\chi \cos \theta + i \sqrt{n^2 \sin^2 \theta - 1}}
\]  
(10) gives the following approximations for the phase shift for grazing angles:
\[
i \ln \mathcal{R} = \pi - \Theta_r
\]
\[
\simeq \pi - \frac{2\chi}{\sqrt{n^2 - 1}} \cos \theta - \frac{\chi(3n^2 - 2\chi^2)}{3(n^2 - 1)^{3/2}} \cos^3 \theta
\]
\[\quad - \frac{\chi(15n^4 - 20n^2\chi^2 + 8\chi^4)}{20(n^2 - 1)^{5/2}} \cos^5 \theta + O(\cos^7 \theta).
\]  
(11)

Nevertheless, direct use of this phase shift in the equations for internal rays, as suggested in \([22]\), leads to incorrect results. The reason is a well-known Goos–Hänchen effect the shift of the reflected beam along the surface. The beams behave as if they are reflected from a fictitious surface shifted from the real boundary at \(\sigma_r = (\Theta_r)/(2k \cos \theta)\). That is why we may substitute the problem for a dielectric body; with the problem for an equivalent body enlarged on \(\sigma_r\) with the totally reflecting boundaries. The parameters of an equivalent spheroid are marked in the following with overbars.

The eikonal equation separates in spheroidal coordinates if we choose \(S = S_1(\xi) + S_2(\eta) + S_3(\phi)\)
\[
\frac{\xi^2 - s}{\xi^2 - \eta^2} \left( \frac{\partial S_1(\xi)}{\partial \xi} \right)^2 + \frac{1 - \eta^2}{\xi^2 - \eta^2} \left( \frac{\partial S_2(\eta)}{\partial \eta} \right)^2
\]
\[+ \left( \frac{\xi^2 - s}{(\xi^2 - s)(1 - \eta^2)} \right) \left( \frac{\partial S_3(\phi)}{\partial \phi} \right)^2 = \frac{n^2 d^2}{4}.
\]  
(12)

After immediate separation of \((\partial S)/(\partial \phi)) = \mu\), we have
\[
(\xi^2 - s) \left( \frac{\partial S_1(\xi)}{\partial \xi} \right)^2 + (1 - \eta^2) \left( \frac{\partial S_2(\eta)}{\partial \eta} \right)^2
\]
\[+ s\mu^2 \frac{1}{\xi^2 - s} - \frac{\mu^2}{1 - \eta^2} \left( \xi^2 - s \right) = 0.
\]  
(13)

Introducing another separation constant \(\nu\), we obtain solutions
\[
\frac{\partial S_1(\xi)}{\partial \xi} = \frac{n^2 d^2 \xi^2}{4(\xi^2 - s) - \xi^2 - s - s\mu^2} \left( \xi^2 - s \right)^{1/2}
\]
\[
\frac{\partial S_2(\eta)}{\partial \eta} = \frac{\nu^2}{1 - \eta^2} - \frac{n^2 d^2 \eta \xi^2}{4(1 - \eta^2) - (1 - \eta^2)^2} \left( \eta^2 - \eta \right)^{1/2}
\]  
(14)

which, after some manipulations, transform to
\[
\frac{\partial S_1(\xi)}{\partial \xi} = \frac{\nu d}{2} \sqrt{(\xi^2 - \eta^2)(\xi^2 - s - \eta^2)} \xi^2 - s
\]
\[
\frac{\partial S_2(\eta)}{\partial \eta} = \frac{\nu d}{2} \sqrt{(\eta^2 - \eta^2)(\xi^2 - s - \eta^2)} \eta^2 - \eta^2
\]
\[
\frac{\partial S_3(\phi)}{\partial \phi} = \mu
\]  
(15)

where
\[
\eta_r^2 = \frac{(1 + s\alpha) - \sqrt{(1 + s\alpha)^2 - 4s\alpha n_0^2}}{2s\alpha}
\]
\[
\xi_r^2 = \frac{(1 + s\alpha) + \sqrt{(1 + s\alpha)^2 - 4s\alpha n_0^2}}{2s\alpha}
\]
\[= \frac{1 + s\alpha}{\alpha} - s\eta_r^2.
\]  
(16)

where \(\alpha = (n^2 d^2)/(4\nu^2)\) and \(\eta_r^2 = 1 - \mu^2/\nu^2\). We now turn to the quasi-classical ray interpretation \([7], [8]\). The equation for the eikonal describes the rays that can spread inside a spheroid along the straight line. These are the rays that freely go inside spheroid, then touch the surface and reflect. For the WGMs, the angle of reflection is close to \(\pi/2\). The closest to the center points of these rays form the caustic surface, which is the ellipsoid determined by a parameter \(\xi_r\). The rays are the tangents to this internal ellipsoid, and follow along the geodesic lines on it. In the case of an ideal sphere, all the rays of the same family lie in the same plane. However, even a slightest eccentricity removes this degeneracy, and inclined closed circular modes (which should be more accurately called quasimodes \([24]\)) are turned into open-ended helices winding up on caustic spheroid precessing \([25]\), and filling up the whole region as in a clew. The upper and lower points of these trajectories determine the other caustic surface with a parameter \(\eta_r\), determining a two-sheeted hyperboloid for prolate or a one-sheeted hyperboloid for oblate spheroid. The value of \(\eta_r\) has a very simple mechanical interpretation. The rays in the eikonal approximation are equivalent to the trajectories of a point-like billiard ball inside the cavity. As the axisymmetrical surface cannot change the angular momentum related to the \(z\) axis, it should be conserved as well as the kinetic energy (velocity). That is why \(\eta_r\) is simply equal to the sine of the angle between the equatorial plane and the trajectory crossing the equator, and at the same time it determines the maximum elongation of the trajectory from the equatorial plane. If all the rays touch the caustic or boundary surface with phases that form a stationary stationery (that means that the phase difference along any closed curve on them is equal to an integer times \(2\pi\)), then the eigenfunction, and hence eigenfrequency, is found.

To find the circular integrals of phases \(kS\) \((15)\), we should take into account the properties of phase evolutions on caustic and reflective boundaries. Every touching of caustic adds \(\pi/2\) (see, for example, \([8]\)) and reflection adds \(\pi\). Thus, for \(S_1\) we have one caustic shift of \(\pi/2\) at \(\xi_r\) and one reflection from the equivalent boundary surface \(\xi_s\) (at the distance \(\sigma\) from the real surface), for \(S_2\) 2 times \(\pi/2\) due to caustic shifts at \(\pm \eta_r\), and we should add nothing for \(S_3\)
\[
k\Delta S_1 = 2k \int_{\xi_r}^{\xi_s} \frac{dS_1}{d\xi} d\xi = 2\pi(q - 1/4)
\]
\[
k\Delta S_2 = 2k \int_{-\eta_r}^{\eta_r} \frac{dS_2}{d\eta} d\eta = 2\pi(p + 1/2)
\]
\[
k\Delta S_3 = k \int_{0}^{2\pi} \frac{dS_3}{d\phi} d\phi = 2\pi|m|
\]  
(17)
where \( q = 1, 2, 3 \ldots \) is the order of the mode, showing the number of the zero of the radial function on the surface, and \( p = l - |m| = 0, 1, 2 \ldots \). These conditions plus integrals (15) completely coincide with those obtained by Bykov [26]–[28] if we transform ellipsoidal to spheroidal coordinates, and have clear geometrical interpretation. The integral for \( S_1 \) corresponds to the difference in lengths of the two geodesic curves on \( \eta_r \) between two points \( P_1 = (\xi, \eta_r, \phi_1) \) and \( P_2 = (\xi, \eta_r, \phi_2) \). The first one goes from the caustic circle of intersection between \( \xi \) and \( \eta_r \) along \( \eta_r \) to the boundary surface \( \xi \), reflects from it, and returns back to the same circle. The second is simply the length of the circle of the intersection of \( \xi \) and \( \eta_r \). These are elliptic integrals. For the WGMs, when \( \eta_r \ll 1 \) and \( \xi_0 - \xi \ll \xi, S_2 \) may be expanded into a series over \( \eta_r \) and \( \xi \) and integrated with the substitutions of 
\[
\eta = \eta_r \sin \psi, \xi = (\xi^2 - \xi_r^2)/\xi_r^2, \quad \text{and, finally, expressing spheroidal coordinates \( \xi \) and expressing \( \xi_0 \) through parameters of spheroid, we have (18), shown at the bottom of the page. Now, we should solve the following system of equations:
\]

\[
k \Delta S_1 = 2k S_1(\zeta_0) \approx \frac{2b^3 nk \bar{a}}{3\bar{a}^3 \sqrt{1 + \zeta_0}} \frac{s}{s_0} \frac{3}{2} \left( 1 - \frac{5a^2 - 2b^2}{5a^2} \zeta_0 - \frac{b^2 - a^2}{2b^2} \eta_r^2 \right) = 2\pi (q - 1/4)
\]

\[
k \Delta S_2 = k S_2(2\pi) \approx \pi \frac{nk b}{\sqrt{1 + \zeta_0}} \eta_r^2 \left( 1 + \frac{a^2 + b^2}{8b^2 - \eta_r^2} \right) = 2\pi (p + 1/2)
\]

\[
k \Delta S_3 = 2\pi k \mu = 2\pi \frac{nk \bar{a}}{\sqrt{1 + \zeta_0}} \sqrt{1 - \frac{\zeta_0(b^2 - \bar{a}^2)}{\bar{a}^2}} \sqrt{1 - \eta_r^2} = 2\pi |m|.
\]

Using the method of sequential iterations, starting, for example, from \( nk^{(0)} = l, \zeta_0^{(0)} = \eta_r^{(0)} = 0 \), this system may be resolved

\[
\eta_r^2 = \frac{(2p + 1)a}{b} - 1 \left[ 1 + \beta \left( \frac{b^2 - a^2}{2b^2} \left( \frac{l}{2} \right)^{-2/3} \right) + O(l^{-2}) \right]
\]

\[
\zeta_0 = -\beta \eta_r a^2 \frac{l^{-2/3}}{l^{-2/3}} \eta_r \left[ 1 - \beta \left( \frac{5a^2 - 3b^2}{5b^2} \left( \frac{l}{2} \right)^{-2/3} \right) + O(l^{-5/3}) \right]
\]

\[
nk \Delta = nk(a - \sigma_r) = l - \beta \left( \frac{l}{2} \right)^{1/3} + \frac{2p(a - b) + a}{2b} - \frac{\chi a}{\sqrt{n^2 - 1}} - \frac{3b^3}{20} \left( \frac{l}{2} \right)^{-1/3} - \frac{\beta \eta_r 2p(a - b) + a^3}{b^3} + \frac{2n \chi (2a^2 - 3a^2)}{(n^2 - 1)3/2} \frac{(l/2)^{-2/3}}{O(l^{-1})}
\]

where, for the convenience of comparison, we introduced \( \beta_q = \frac{[(3/2)\pi(q - 1/4)]^{2/3}}{2} \). The value of \( \cos \theta \) needed for the calculation of \( \Theta \) (12) may be estimated as \( \cos \theta = \sqrt{1 - l^2/(nk \Delta)} \approx \beta \theta^{-1/3} \). The first three terms for \( nk \Delta \) were obtained in [3], [26]–[28] from different considerations, the last three of which are new.

To test this series, we calculated using finite-element method (FEM) eigenfrequencies of TE modes in spheroids with different

\[
S_1 = \frac{nb^3}{2a^2 \sqrt{1 + \zeta_0}} \int \sqrt{\zeta^2 + 1 + \zeta_0} \sqrt{1 + \zeta - \eta_r^2(1 + \zeta_0)(b^2 - a^2)/b^2} (1 + \zeta_0 + (\zeta - \zeta_0)b^2/a^2) \sqrt{1 + \zeta_0} d\zeta
\]

\[
= \frac{nb^3}{2a^2 \sqrt{1 + \zeta_0}} \left[ \frac{2}{3} \zeta^{3/2} - \frac{10a^2 - 4b^2}{15a^2} \zeta^{5/2} + \frac{a^2 - b^2}{3b^2} \zeta^{3/2} \eta_r^2 \right] + O \left( \zeta^{7/2}, \eta_r^{5/2}, \eta_r^{7/2} \right)
\]

\[
S_2 = \frac{nd \zeta}{2 \eta_r^2} \int \frac{\cos^2 \psi \sqrt{\zeta^2 - s^2} \sin^2 \psi}{1 - \eta_r^2 \sin^2 \psi} d\psi
\]

\[
= \frac{nd \zeta}{2 \eta_r^2} \left[ \frac{2 \psi + \sin 2\psi}{4} + \frac{(2\zeta^2 - s)(4\psi - \sin 4\psi)}{64\zeta^2} \eta_r^2 + \frac{8\psi^4 - 4f\zeta^4 - 1)(12\psi + \sin 6\psi - 3 \sin 4\psi - 3 \sin 2\psi)}{1536\zeta^4} \right] + O \left( \eta_r^4 \right)
\]

\[
S_3 = \mu \phi
\]
ecentricities and totally reflecting boundaries for \( l = m = 100 \) (Fig. 2). Significant improvement of our series is evident. The divergence of the series for large eccentricities is explained by the fact that the approximation that we used to calculate the divergence of the series for large eccentricities is explained by (Fig. 2). Significant improvement of our series is evident. The eccentricities and totally reflecting boundaries for spheroid.

![Fig. 2. Comparison of the precision of calculation of eigenfrequencies in a spheroid.](image)

If we put \( a = b \), then all six terms in the obtained series coincide with that obtained in [9] from the exact solution in a sphere with two minor differences: 1) Airy function zeroes \( \alpha_q \approx (-2.3381, -4.0879, -5.5206, \ldots) \) stand in Schiller’s series instead of approximate \( \beta_q \) values \( (\alpha_q - \beta_q \approx -0.017; -0.0061; -0.0033, \ldots) \). The reason is that the eikonal approximation breaks down on caustic, where its more accurate extension with Airy functions is finite [7]. To make our solution even better, we may just formally using \( \alpha_q \) instead of \( \beta_q \), hence obtaining the final formula (1); and 2), a minor difference in the last term is caused, we think, by a misprint in [9], where in our designations, instead of \( d_2 = -2^{2/3} / a^2 (-3 + 2 \chi^2) / \alpha_q / 6 \), should be \( d_2 = -2^{2/3} (-3n^2 + 2 \chi^2) / \). The eikonal equation for the sphere may be solved explicitly, and the expansion of the solution shows that quasi-classical approximation breaks down on a term \( O(l^{-1}) \), and of the same order should be the error introduced with substitution of vector equations by scalar ones.

We may now calculate the dependence of mode separation on three indices up to \( O(l^{-2}) \)

\[
\frac{1}{\omega} \frac{\partial \omega}{\partial l} \approx l^{-1} \left[ 1 + \frac{\alpha_q}{6} \left( \frac{l}{2} \right)^{-2/3} \right]
\]

\[
\frac{1}{\omega} \frac{\partial \omega}{\partial m} \approx l^{-1} \frac{b-a}{b} \left[ 1 + \frac{\alpha_q}{12} \frac{(b-a)(a+2b)}{b^2} \right] \left( \frac{l}{2} \right)^{-2/3}
\]

\[
\frac{1}{\omega} \frac{\partial \omega}{\partial \eta} \approx \frac{\pi}{2\sqrt{-\alpha_q}} \left( \frac{l}{2} \right)^{-2/3} \left[ 1 + \frac{\alpha_q}{20} \left( \frac{l}{2} \right)^{-2/3} \right].
\] (21)

It is interesting to note that when \( a = 2b \) (oblative spheroid with eccentricity \( \varepsilon = \sqrt{0.75} \)), the eigenfrequency separation in the first order of approximation between modes with the same \( l \) becomes equal to the separation between modes with different \( l \) and the same \( l - m \) (free spectral range). The difference appears only in the term proportional to \( O(l^{-2/3}) \). This situation is close to the case that was experimentally observed in [3]. This new degeneracy has a simple quasigeometrical interpretation as in the case of a sphere, geodesic lines inclined to the equator plane on such a spheroid are closed curves returning at the same point of the equator after the whole revolution; crossing, however, the equator not twice as big circles on a sphere, but four times.

IV. ARBITRARY CONVEX BODY OF REVOLUTION

To find eigenfrequencies of whispering gallery modes in an arbitrary body of revolution, one may directly use the results of Section III by fitting the shape of the body in a convex equatorial area by an equivalent spheroid. In fact, a body should be convex only in the vicinity of the WG mode itself. For example, a torus with a circle of radius \( r_0 \) with its center at a distance \( R_0 \) from the z axis as a generatrix may be approximated by a spheroid with \( a = R_0 + r_0 \) and \( b = \sqrt{(R_0 + r_0)R_0} \). Nevertheless, a more rigorous approach may be developed.

The first step is to find families of caustic surfaces. This is not a trivial task in the general noninvertible case. The following approximation may be used to find the first family of caustic surfaces [8] which are the place of biinvolute curves (the difference in length between a sum of two tangent lines from a point on a curve to a biinvolute curve and an arc between these lines is constant) for geodesic lines characteristic for a chosen WGM mode on the surface:

\[
\sigma_c(s) = -\frac{1}{2} \kappa^2 r_k^{1/3} \left( \frac{\alpha_q}{6} \right) + O(\kappa^4)
\]

\[
\cos \theta(s) = \kappa r_k^{1/3} \left( \frac{\alpha_q}{6} \right)
\] (22)

where \( \sigma_c(s) \) is the normal distance from a point \( s \) on the surface of the body to a caustic surface, \( \kappa \) is a parameter of a family, and \( r_k \) is the radius of curvature of the geodesic line (curvature of the surface in the direction of the beam) and \( \cos \theta \) is angle of incidence of the beam in the point \( s \).

We have found a caustic surface from the first family, parametrized as

\[
z = u
\]

\[
x = g(u) \cos \phi
\]

\[
y = g(u) \sin \phi.
\] (23)

A geodesic line for this surface is given by the following integral:

\[
\frac{d\phi}{du} = c_1 \frac{\sqrt{1 + g'^2}}{g(u) \sqrt{g'^2(u) - c_1^2}}
\] (24)

where \( c_1 \) is some constant, which is equal, in our case, to \( \rho_c = g(z_{\text{max}}) \)-the radius of the caustic circle at maximum distance.
from the equatorial plane. The length of geodesic line is

$$\frac{ds}{du} = \sqrt{1 + g^2 + 2g^2 \left( \frac{d\phi}{du} \right)^2} = \frac{g\sqrt{1 + g^2}}{\sqrt{g^2 - \rho_0^2}}. \quad (25)$$

The length of geodesic line connecting points $\phi_1$ and $\phi_2$ is

$$L_1^s = 2\int_{-u_0}^{u_0} \frac{g\sqrt{1 + g^2}}{\sqrt{g^2 - \rho_0^2}} du. \quad (26)$$

The length of the arc from $\phi_0 = 0$ to $\phi_2 = 2\int_{\eta_0}^{\eta_2} (d\phi)/(du) du$ is equal to $L_2^s = \rho \rho \phi$.

$$L_2^s = 2\int_{-u_0}^{u_0} \rho \rho_2^2 \frac{\sqrt{1 + g^2}}{\sqrt{g^2 - \rho_0^2}} du. \quad (27)$$

Finally,

$$nk(L_1^s - L_2^s) = 2nk\int_{-u_0}^{u_0} \frac{\sqrt{1 + g^2}}{g} \frac{\sqrt{g^2 - \rho_0^2}}{4} du = 2n(p + 1/2). \quad (28)$$

In an analogous way, for another geodesic line on a caustic surface from the other family,

$$z = v$$
$$x = h(v) \cos \phi$$
$$y = h(v) \sin \phi$$

and we have

$$nk(L_1^b - L_2^b) = 2nk\int_{v_0}^{v_0} \frac{\sqrt{1 + h^2}}{h} \frac{\sqrt{h^2 - \rho_0^2}}{4} dv = 2\pi(q - 1/4). \quad (30)$$

The third condition is

$$2\pi nk \rho_0 = 2\pi |n|. \quad (31)$$

With the substitution $u = (d/2)\xi, v = (d/2)\xi, g(u) = (d/2)\sqrt{\xi^2 - \xi^2} h(u) = (d/2)\sqrt{\xi^2 - \xi^2}$, we obtain expressions for the spheroid obtained previously. For a torus, caustic surfaces are approximately toruses and cones determined by the equations

$$g(u) = R_0 + \sqrt{r_c^2 - u^2}$$
$$h(v) = R_0 + v \frac{\rho - R_0}{\sqrt{a^2 - (\rho - R_0)^2}}$$

Another method based on the Wentzel–Kramers–Brillouin (WKB) approach for the calculation of eigenfrequencies of WGMs in arbitrary shaped bodies of rotation was proposed recently [29]. A similar technique was used in [30]. It would be interesting to compare the precision of both approaches, but as there are no explicit solutions of common problems obtained with both methods, this comparison is not possible. However, judging the known approximations obtained with the WKB method for spheroidal functions [14], [19], [20], we consider that our method is more precise and general.

V. CONCLUSION

In conclusion, we have analyzed a quasi-classical method of calculation of eigenfrequencies in spheroidal cavities, and found that it gives approximations correct up to the term proportional to $l^{-2/3}$. This method may be easily expanded on arbitrary convex bodies of revolution.

REFERENCES


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