Analysis of parametric oscillatory instability in Fabry–Perot cavity with Gauss and Laguerre–Gauss main mode profile

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A B S T R A C T

We calculate the parametric instabilities in Fabry–Perot cavities of advanced VIRGO and LIGO interferometers with different main mode profiles. All unstable combinations of elastic and Stokes modes both for the case with TEM00 and LG33 as a carriers are deduced.

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1. Introduction

Sensitivity increase of gravitational wave detectors like advanced VIRGO and LIGO is expected to be obtained by rising the value of power stored in the Fabry–Perot (FP) resonator optical mode. However, high values of circulating power may be a source of the nonlinear effects which will prevent from reaching the projected sensitivity. It is appropriate to remind that nonlinear coupling of elastic and light waves in continuous media produces Mandelstam–Brillouin scattering. It is a classical parametric effect; it is often explained in terms of quantum physics: one quantum \( h\omega_0 \) of main optical wave transforms into two, i.e., \( h\omega_1 \) in the additional optical wave (Stokes wave: \( \omega_1 < \omega_0 \)) and \( h\omega_m \) in the elastic wave so that \( \omega_0 = \omega_1 + \omega_m \). The irradiation into the anti-Stokes wave is also possible \( \omega_m = \omega_0 - \omega_1 \), however, in this case the part of energy is taken from the elastic wave. The physical mechanism of this coupling is the dependence of refractive index on density which is modulated by elastic waves. If the main wave power is large enough the stimulated scattering will take place, the amplitudes of elastic and Stokes waves will increase substantially.

In gravitational wave detectors elastic oscillations in FP resonator mirrors will interact with optical ones being coupled parametrically due to the boundary conditions on the one hand, and due to the ponderomotive force on the other hand. Two optical modes may play roles of the main and Stokes waves. High quality factors of these modes and of the elastic one will increase the effectiveness of the interaction between them and may give birth to the parametric oscillatory instability which is similar to stimulated Mandelstam–Brillouin effect. This undesirable effect of parametric oscillatory instability may create a specific upper limit for the value of power \( W_c \) circulating inside the cavity [1].

It is interesting that the effect of parametric instability is important not only for large scale gravitational-wave interferometers. Recently, the instability produced by optical rigidity was observed in experiment [2]. K. Vahala with collaborators has also observed it in microscale whispering gallery optical resonators [3,4]. Zhao et al. [5] have shown in experiment, using an 80 m Fabry–Perot cavity, that parametric instability effect can indeed occur. The experimental results are in good agreement with theoretical predictions [1].

The condition of parametric instability for Fabry–Perot cavity may be written in simple form [1]:

\[
R = \frac{A_1 W_0 \omega_1}{c L m \omega_m \gamma \gamma'} \frac{1}{1 + \frac{\Delta^2}{\gamma^2}} > 1, \quad \gamma \gg \gamma_m.
\]

(1)

\[
A_1 = \frac{V (\int |A_0(r_\perp)| A_1(r_\perp) u_2 \, dr_\perp)^2}{\int |A_0|^2 \, dr_\perp \int |A_1|^2 \, dr_\perp \int |\vec{u}|^2 \, dV},
\]

(2)

where \( c \) is a speed of light, \( L \) is a distance between the FP mirrors, \( m \) is a mirror’s mass, \( \gamma_m, \gamma' \) are the relaxation rates of elastic and Stokes modes correspondingly, \( W \) is a power circulating inside the cavity. \( A_1 \) is an overlapping factor of elastic and optical modes and \( \Delta = \omega_0 - \omega_1 - \omega_m \) is a detuning value. Here \( A_0 \) and \( A_1 \) are the functions of the distributions over the mirror surface of the optical fields in the main and Stokes optical modes correspondingly, vector \( \vec{u} \) is the spatial vector of displacements in elastic mode, \( u_2 \) is the component of \( \vec{u} \), normal to the mirror’s surface, \( \int dr_\perp \) corresponds to the integration over the mirror surface and \( \int dV \) – over the mirror volume \( V \).

As we can see from formula (1) the parametric instability is a threshold effect and it takes place if optical power \( W \) in main
In this case elastic modes and overlapping factor and we see that in this case full depression of parametric instability that appropriate anti-Stokes mode exists is extremely small the same overlapping factors:

and anti-Stokes modes may be analytically calculated for Gaussian detail to avoid it. In particular, we need detailed information about evalutional wave detectors and we need to know this “enemy” in this particular case (in opposite case the overlap factor has to depend on distance between the center of mirror and center of main optical mode distribution over mirror surface).

Fig. 1. Schematic structure of optical Laguerre–Gauss modes in the FP cavity. The modes of the main frequencies sequence are shown by higher peaks. It is shown that Stokes mode with frequency \( \omega_1 \) may not have suitable anti-Stokes mode (it is denoted by question-mark).

mode of Fabry–Perot cavity is bigger than the threshold power \( W_c \).

The value \( W_c \) depends on the detuning \( \Delta \) between optical and elastic modes and overlapping factor \( A_1 \), in particular, the \( W_c \) has minimum if detuning is less than relaxation rate of Stokes mode \( \Delta < \gamma_1 \) and overlapping factor is large enough.

Analyzing this equation D’Ambrosio and Kells [6] pointed out that the presence of the anti-Stokes mode can considerably depress or even exclude parametric instability. Indeed, the condition of parametric oscillatory instability in the presence of Stokes mode with frequency \( \omega_{01} = \omega_0 + \omega_m \) has the following form [7]:

\[
\frac{A_1W_1\omega_1}{cLm_1H_1n_1} \times \frac{1}{1 + \frac{\Delta_1^2}{\gamma_1^2}} - \frac{A_{1a}W_1\omega_{a1}}{cLm_1H_1n_1} \times \frac{\omega_{a1} \gamma_1}{\omega_1 \gamma_1} \times \frac{1}{1 + \frac{\Delta_1^2}{\gamma_1^2}} > 1,
\]

where \( \gamma_1 \) is a relaxation rate of anti-Stokes mode and \( \Delta_1 = \omega_{a1} - \omega_0 - \omega_m \) is a detuning value.

For example, let the main, Stokes and anti-Stokes modes be equidistant and belong to the main frequency sequence \( \omega_1 = \pi (K - 1)c/L, \omega_0 = \pi Kc/L, \omega_{a1} = \pi (K + 1)c/L \) (K is an integer). In this case \( \Delta = \Delta_1 \), the main, Stokes and anti-Stokes modes have the same Gaussian distribution over the cross section and hence the same overlapping factors: \( A_1 = A_{1a} \). It means that the second term in the right part of equation is larger than first term, the positive damping introduced into elastic mode by the anti-Stokes mode is greater than negative damping due to the Stokes mode, hence the parametric instability is impossible. This case has been analyzed in details in [6].

For the case when the Stokes and anti-Stokes modes do not belong to the main sequence (non-zero radial \( N \) and azimuth \( M \) numbers) the frequencies of the suitable Stokes and anti-Stokes modes are not equidistant from the main mode \( (\Delta \neq \Delta_1) \) and have different spatial distributions \( (A_1 \neq A_{1a}) \). It takes place when main mode is TEM\(_{00}\) and Fig. 1 illustrates it. For the shown Stokes mode (left to the main mode) there is no suitable anti-Stokes mode (it should be located right to the main one). The probability that appropriate anti-Stokes mode exists is extremely small and we see that in this case full depression of parametric instability does not take place. However, when main mode has non-zero radial and azimuth numbers (high order Laguerre–Gauss modes LG\(_{m,n}\)) the possibility of parametric instability depression by existence of suitable anti-Stokes mode will increase.

Parametric instability is a serious problem for advanced gravitational wave detectors and we need to know this “enemy” in detail to avoid it. In particular, we need detailed information about pairs of Stokes modes and elastic modes which may be possible candidates for parametric instability. It is known that Stokes and anti-Stokes modes may be analytically calculated for Gaussian beams. In contrast, elastic modes may be calculated numerically (some method of analytical elastic modes calculation has been proposed in [8]). Hence, the accuracy of parametric instability forecast (and success of methods to prevent it) directly depends on how accurately we can calculate normal (eigen) frequencies and spatial distributions of elastic modes. Using ANSYS® code, for example, we can obtain the accuracy of elastic modes calculations of about 0.5% or poorer [9–12].

In [13,14] the importance of accurate numerical calculation of elastic modes in the mirrors of advanced LIGO was discussed to enable precise predictions of the problem of parametric oscillatory instability. In [13] it has been proposed accuracy estimations through use of analytical solutions based on Chree–Lamb modes. Small deviations from cylindrical test mass shape may produce splitting of non-axial symmetric elastic modes into doublets. This splitting may increase the possibility of parametric oscillatory instability too [14].

It is known the important goal in gravitational wave detectors is to minimize fundamental and technical noise contribution. For example, progressive way to lower the thermal noise is to change the mode shape of the laser beam inside the interferometer. It is worth noting that different shapes of beam have been proposed for reducing thermal noise such as mesa beams [15–17], conical modes [18] and high order Laguerre–Gauss modes [19]. At present, there are some techniques for the generation of high order LG modes using holograms [20,21], gratings [22] and mode transformers [23, 24] with high conversion efficiency [22,25]. In [26] using numerical interferometer simulations the comparison of behavior of the LG\(_{33}\) mode with the fundamental mode TEM\(_{00}\) are discussed. The LG\(_{33}\) mode performs similar if not even better than commonly used TEM\(_{00}\) for all considered aspects of interferometric sensing.

In this Letter we estimate the possibility of parametric instabilities in Fabry–Perot cavity of advanced VIRGO and LIGO interferometers both for Gauss TEM\(_{00}\) and Laguerre–Gauss LG\(_{33}\) modes as a carriers [26]. In Section 2 we provide estimations of PI in Fabry–Perot cavity of advanced VIRGO and LIGO interferometers and discuss our results, in Section 3 we make some useful conclusions for future research.

2. Results

In order to predict the unstable combinations of Stokes and elastic modes we have to take into account the additional azimuth numbers condition of parametric instability. For example, for TEM\(_{00}\) as a carrier (when the elastic modes have azimuth dependence \( e^{im\phi} \) and Stokes modes have \( e^{in\phi} \) the non-zero overlap factor will be if \( m = n \) (see the formula (2) for \( A_1 \)). In turn, if we use LG\(_{33}\) as a carrier the non-zero overlap factor will be if \( |m| = |n| = 3 \). It is worth noting that it is correct only if cylinder center coincides with laser spot center and below we consider this particular case (in opposite case the overlap factor has to depend on distance between the center of mirror and center of main optical mode distribution over mirror surface).

Table 1

<table>
<thead>
<tr>
<th>VIRGO</th>
<th>LIGO</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W = 0.76 \times 10^8 ) W</td>
<td>( W = 0.83 \times 10^8 ) W</td>
</tr>
<tr>
<td>( L = 3000 ) m</td>
<td>( L = 4000 ) m</td>
</tr>
<tr>
<td>( T = 7 \times 10^{-3} )</td>
<td>( T = 14 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \lambda = 1064 ) nm</td>
<td>( \lambda = 1064 ) nm</td>
</tr>
<tr>
<td>( m = 40 ) kg</td>
<td>( m = 40 ) kg</td>
</tr>
<tr>
<td>( r = 0.17 ) m</td>
<td>( r = 0.17 ) m</td>
</tr>
<tr>
<td>( H = 0.2 ) m</td>
<td>( H = 0.2 ) m</td>
</tr>
<tr>
<td>( \gamma_1 = 6 \times 10^{-3} ) s(^{-1} )</td>
<td>( \gamma_1 = 6 \times 10^{-3} ) s(^{-1} )</td>
</tr>
<tr>
<td>( \gamma = c/T_{14} = 175 ) s(^{-1} )</td>
<td>( \gamma = c/T_{14} = 262.5 ) s(^{-1} )</td>
</tr>
</tbody>
</table>

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Table 2  
Unstable combination of elastic and Stokes optical modes in FP cavity of VIRGO configuration with LG33 as a carrier. In the case of TEM00 carrier we did not find any unstable modes in given frequency range due to small overlap factors and large detuning values Δ.

<table>
<thead>
<tr>
<th>ωm/2π, Hz</th>
<th>m</th>
<th>Stokes optical mode</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>38,577</td>
<td>1</td>
<td>LG12</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Fig. 2. Displacement vector component distribution $u_z$ of elastic mode with frequency 38,577 Hz and azimuth index $m = 1$. The numerical calculations of cylinder have been made on triangle mesh with about 40,000 meshing elements.

For Fabry–Perot cavity we use the parameters of advanced Virgo and LIGO interferometers shown in Table 1 where $T$, $λ$, $r$, $H$ are the transmittances of input FP mirrors, $λ$ is a wave length of input laser and $r$, $H$ are radius and height of cylindrical fused silica mirror. Note that the condition of parametric instability in Michelson interferometer with Fabry–Perot cavities in the arms with additional power and signal recycling mirrors (configurations of advanced LIGO and Virgo interferometers) and in alone Fabry–Perot cavity can substantially differ – see discussion in Section 3.

The calculation of parametric instabilities is very sensitive to small changes in the model parameters and gives only statistical result for number of unstable modes. In particular, uncertainties in the mirror radius of curvature and small variations in materials and temperature are sufficient to move the optical and mechanical resonances of the cavities. We estimate the number of unstable combinations of elastic and Stokes modes both for the case with TEM00 as a carrier (TEM00 case) and LG33 as a carrier (LG33 case). The elastic modes were calculated numerically using COMSOL® code on triangle mesh with about 40,000 meshing elements. We restrict our estimates by elastic frequency range up to $2π \times 40,000$ s$^{-1}$ because, on the one hand, the more the elastic frequency the less the parametric gain. On the other hand, high elastic mode distributions are difficult to determine exactly among other elastic modes due to degradation of numerical accuracy of COMSOL® code.

For Virgo configuration we use Fabry–Perot cavities with laser spot radii on mirrors surfaces $r = 6.47$ cm and $w = 3.94$ cm in TEM00 and LG33 cases correspondingly [26]. The waist radii are situated in the middle of the cavities and are equal to $w_0 \approx 0.79$ cm and $w_0 \approx 1.38$ cm for these cases. Level of diffractional losses in clipping approximation is hold to be $l_{clip} = 1$ ppm. For Ligo configuration we use Fabry–Perot cavities with laser spot radii on mirrors surfaces $w = 6$ cm and $w = 3.94$ cm in TEM00 and LG33 cases correspondingly [27]. The waist radii are situated in the middle of the cavities and are equal to $w_0 \approx 1.15$ cm and $w_0 \approx 1.99$ cm for these cases, and level of diffractional losses in clipping approximation is hold to be $l_{clip} = 1$ ppm or less.

We calculate the values of overlap factors for all combinations of elastic and optical modes (up to 9th order) but, on the other hand, we use azimuth numbers condition in each case. It is also worth noting that the analysis of parametric instability in each case has been realized with taking into account the appropriate anti-Stokes modes which may suppress the parametric instabilities. We also took into account the diffractional losses of Stokes and anti-Stokes modes in clipping approximation for estimations of optical modes relaxation rates as it has been done in [7].

In Table 2 all unstable combinations of elastic and Stokes modes and parametric gains $R$ for LG33 case in Virgo interferometer are shown. There is only one unstable combination and there is not any anti-Stokes mode that can suppress the parametric instability. In Fig. 2 we can see the z-component displacement vector distribution for this unstable elastic mode. In the case of TEM00 carrier we did not find any unstable modes in given frequency range due to small overlap factors and large detuning values Δ.

In Table 3 all unstable combinations of elastic and Stokes modes and parametric gains $R$ for LG33 case in LIGO interferometer are shown. There are three unstable combinations. In Fig. 3 we present the z-component displacement vector distributions for these elastic modes. In the case of TEM00 carrier we found only one unstable mode in given frequency range which is shown in Table 4.

We can see that possibility of parametric instability in given elastic frequency range for LG33 case will be slightly larger than for the TEM00 case both in Fabry–Perot cavities of VIRGO and LIGO interferometers.

3. Conclusions

In this Letter we deduced the analysis of parametric oscillatory instability in FP cavities of gravitational wave detectors like advanced LIGO and Virgo in the model with cylindrical mirrors without flats and suspension ears. The main goal of our estimations was to compare a difference in TEM00/LG33 behavior be-
between advanced LIGO and VIRGO Fabry–Perot cavities for parametric instabilities in such simple model. The number of obtained unstable modes for LG33 case is slightly larger than for TEM00 case in elastic modes frequency range up to 40 kHz, however, this difference is not large enough to conclude that LG33 case is more dangerous than TEM00 case. No doubts, there is necessity to perform detailed analysis for full scale schemes of advanced LIGO and VIRGO in LG33 case with wide elastic modes frequency range, very high accuracy of elastic modes calculations [8] and mirror flats and ears.

At the end of the Letter we would like to summarize some dangerous points that must be taken into account at present and in future parametric instability research.

It is worth noting that for combinations of modes suitable for parametric instability the overlapping factor $A_1$ may be zero (for example, elastic mode and the Stokes mode can have different dependence on azimuth angle) – we did not mention such combinations in our estimations. However, it is important to take into account that only the elastic mode is attached to the mirror axis in contrast to the optical mode which can be shifted from the mirror axis due to non-perfect optical alignment. Hence, the overlapping factor has to depend on distance $Z$ between the center of mirror and the center of the main optical mode distribution over the mirror surface. It means that $A_1$ may be zero for $Z = 0$ but non-zero for $Z \neq 0$. Therefore, the numerical analysis of the mode structure should evidently include the case when $Z \neq 0$. Note that there is a proposal to use special shift $Z$ of the laser beam of about several centimeters from the mirror axis in order to decrease thermal suspension noise [28].

Imperfections from the cylinder shape (such as flats and suspension ears), should cause splitting of elastic modes with azimuth index $m \geq 1$ into doublets and the difference between doublet frequencies may be large enough or greater as compared with relaxation rate of Stokes mode. Both the appearance of doublets and high density of elastic modes may increase the possibility of parametric instability in advanced VIRGO and LIGO [13,14].

The effect of parametric instability for power recycled interferometer may be larger than for the separate Fabry–Perot cavity because the Stokes mode emitted from the Fabry–Perot cavity throughout its input mirror is not lost irreversible but returns back due to power recycling mirror, therefore, its interaction is prolonged [7]. As it was demonstrated in [29] the presence of power recycling cavity causes parametric gains $R$ to be hugely amplified at $\Lambda = 0$ and reduced by a factor of 2 in off-resonance intervals as compared to an interferometer without power recycling. The detailed analysis of parametric instability in signal recycled interferometer has been realized in [30–32]. On the other hand, the parametric instability in the interferometer with power recycling mirror may be depressed by variation of distance between power recycling mirror and beam splitter [30–33]. Detail analysis of parametric instability in full scale advanced LIGO interferometer has been performed in [33,34] and it will be useful to present the same analysis for LG33 case in future researches.

We think that parametric oscillatory instability effect can be overcome in laser gravitational detectors after developing strategies for its suppression using these investigations.

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